UNCLASSIFIED

AD NUMBER
AD469173
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; AUG 1965. Other requests shall be referred to Office of Naval Research, Washington, DC 20360.**
AUTHORITY
onr ltr 28 jul 1977

THIS PAGE IS UNCLASSIFIED

Some Inequalities for Reliability Functions

by

Z. W. Birnbaum

University of Washington

and

J. D. Esary

Boeing Scientific Research Laboratories

Technical Report No. 44

August 18, 1965

Contract N-onr-477(11)

Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle, Washington

This research was supported by the Office of Naval Research.

SECURITY MARKING

The classified or limited status of this report applies to each page, unless otherwise marked.
Separate page printouts MUST be marked accordingly.

THIS DOCUMENT CONTAINS INFORMATION AFFECTING THE NATIONAL DEFENSE OF THE UNITED STATES WITHIN THE MEANING OF THE ESPIONAGE LAWS, TITLE 18, U.S.C., SECTIONS 793 AND 794. THE TRANSMISSION OR THE REVELATION OF ITS CONTENTS IN ANY MANNER TO AN UNAUTHORIZED PERSON IS PROHIBITED BY LAW.

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

Some Inequalities for Reliability Functions

by

Z. W. Birnbaum

University of Washington

and

J. D. Esary

Boeing Scientific Research Laboratories

Technical Report No. 44

August 18, 1965

Contract N-onr-477(11)

Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle, Washington

This research was supported by the Office of Naval Research.

SOME INEQUALITIES FOR RELIABILITY FUNCTIONS

by

Z. W. Birnbaum
University of Washington

and

J. D. Esary
Boeing Scientific Research Laboratories

1. Basic Concepts and Definitions

1.1 With the advent of very complex engineering designs such as those of high-speed computers or supersonic aircraft, it has become increasingly important to study the relationship between the functioning and failure of single components and the performance of the entire system and, in particular, to be able to make quantitative statements about the probability that the system will perform according to specifications. It is the aim of this paper to present some inequalities for this probability.

1.2 We shall assume that there are only two states possible for every component of a system, as well as for the system itself: either it functions or it fails. When the system consists of n components, we shall ascribe to each of them a binary variable which will indicate its state

$$x_i = \begin{cases} 1 & \text{when the } i\text{-th component functions} \\ 0 & \text{when the } i\text{-th component fails} \end{cases}$$

for $i = 1, 2, \ldots, n$. Similarly we ascribe to the entire system a binary indicator variable

$$u = \begin{cases} 1 & \text{when the system functions} \\ 0 & \text{when the system fails.} \end{cases}$$

When the design of a system is known, then the states of all n components i.e., the values of $x_1, x_2, \cdots, x_n$ determine the state of the system, that is the value of u so that

$$u = \phi\,(x_1, x_2, \cdots, x_n)$$

where ϕ is a function assuming the values 0 or 1. This function ϕ will be called the structure function of the system. The indicator variable x_i will sometimes be referred to as "component x_i" and ϕ will sometimes be called "structure ϕ". The n-tuple of 0's or 1's

$$(x_1, x_2, \cdots, x_n) = \underline{x}$$

will be called the vector of component states or, in short, the "state vector". It can assume any one of the 2^n values represented by the vertices of the unit-cube in n-dimensional space: (0, 0, ..., 0), (1, 0, ..., 0), (1, 1, ..., 0), ..., (1, 1, ..., 1). This set of all possible values of $\underline{x}$ will be denoted by I_n. Thus the structure function $\phi(x_1, x_2, \cdots, x_n) = \phi\,(\underline{x})$ is a binary function on I_n.

1.3 We shall furthermore assume that the state of each component is decided by chance, so that the value actually assumed by x_i is a binary random variable X_i with the probability distribution

$$\left.\begin{aligned} \Pr\{X_i = 1\} &= p_i \\ \Pr\{X_i = 0\} &= q_i = 1 - p_i \end{aligned}\right\} \quad i = 1, 2, \cdots, n$$

and we shall make the assumption that $X_1, X_2, \cdots, X_n$ are

totally independent. The probability p_i will be called the <u>reliability of the i-th component</u>.

In the following it will always be assumed that

(1.3.1) $p_1 = p_2 = \cdots = p_n = p$

that is, that all components have the same reliability, e.g. the reliability of the least reliable component.

1.4 For a known structure function $\phi(\underline{X})$ the value of p determines the probability that the system will function

(1.4.1) $\Pr\{\phi(\underline{X}) = 1 \mid p\} = h_\phi(p)$

i.e., the <u>reliability of the system</u> for given component reliability p. The function $h_\phi(p)$ is called the <u>reliability function</u> for ϕ; we will mostly denote it by $h(p)$ omitting the subscript ϕ.

In this paper we shall present inequalities for $h'(p)$, the derivative of the reliability function, which can be obtained when only partial information about $\phi(\underline{x})$ is available. We shall then discuss a procedure by which such inequalities can be used to obtain some conclusions about $h(p)$.

1.5 The assumption of 1.2 is restrictive since it precludes consideration of systems whose components may function only partially and yet the systems will deliver a satisfactory performance. Similarly, the assumption of 1.3 is rather special, since often functioning or failure of different components of a system is correlated. Nevertheless, these two assumptions are a reasonable approximation to many

practical situations, and they make it possible to simplify the theory to a manageable level.

1.6 For state vectors $\underline{x}$, $\underline{y}$ we shall use the following notations:

$$\underline{x} \leq \underline{y} \quad \text{when} \quad x_i \leq y_i \quad \text{for} \quad i = 1, 2, \cdots, n$$
$$\underline{x} < \underline{y} \quad \text{when} \quad \underline{x} \leq \underline{y} \quad \text{and} \quad x_j < y_j \quad \text{for some} \quad j$$
$$(0_k, \underline{x}) = (x_1, x_2, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n)$$
$$(1_k, \underline{x}) = (x_1, x_2, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_n)$$
$$\underline{0} = (0, 0, \cdots, 0) \; , \quad \underline{1} = (1, 1, \cdots, 1) \; .$$

A component x_k is called <u>essential</u> for the structure $\Phi(\underline{x})$ if there exists a state vector $\underline{x}^*$ such that

$$\Phi\,(1_k, \underline{x}^*) \neq \Phi\,(0_k, \underline{x}^*)$$

If $\Phi(x)$ is any function on I_n , not necessarily a structure function, the same definition of an essential component can be used.

1.7 For given n there are $2^{(2^n)}$ possible different structure functions. Among all these possible structure functions we shall single out the class of <u>coherent</u> structure functions which is defined in the following, (this definition was introduced in [1]) and which not only has some intuitive appeal but also has been found to have a number of rather interesting properties [2].

A structure function $\Phi(\underline{x})$ is coherent when it fulfills the following conditions:

(1.7.1) $\quad \phi(\underline{x}) \leq \phi(\underline{y})$ for $\underline{x} \leq \underline{y}$

(1.7.2) $\quad \phi(\underline{0}) = 0 , \quad \phi(\underline{1}) = 1 .$

From now on we shall assume that all structure functions considered are coherent.

1.8 A state vector $\underline{x}$ is called a path for ϕ when

$$\phi(\underline{x}) = 1$$

and $\underline{x}$ is called a cut for ϕ when

$$\phi(x) = 0 .$$

This terminology is analogous to that used in circuit theory.

From ϕ being coherent follows immediately that

(a) if $\underline{x}$ is a path for ϕ and $\underline{x} \leq \underline{y}$, then $\underline{y}$ is a path for ϕ ,

(b) if $\underline{x}$ is a cut for ϕ and $\underline{x} \geq \underline{y}$, then $\underline{y}$ is a cut for ϕ .

For every state vector $\underline{x} \in I_n$ we define

$$S(\underline{x}) = \sum_{i=1}^{n} x_i = \text{number of functioning components in}$$

$\underline{x}$ and call $S(\underline{x})$ the size of $\underline{x}$.

For a given structure $\phi(\underline{x})$ we consider the following numbers:

(1.8.1) A_j = number of paths of size j , for $j = 0, 1, 2, \ldots, n$

Obviously one has

(1.8.2) $A_j \leq \binom{n}{j}$, $j = 0, 1, \ldots, n$.

1.9 For $\Phi(\underline{x})$ coherent one can prove [1] that

(1.9.1) $\dfrac{A_j}{\binom{n}{j}} \leq \dfrac{A_{j+1}}{\binom{n+1}{j}}$ for $j = 0, 1, \cdots, n-1$

(1.9.2) $h(0) = 0, \quad h(1) = 1$

(1.9.3) $h'(p) > 0$ for $0 < p < 1$.

2. Inequalities for $h'(p)$ and grids for $h(p)$

2.1 Let us assume that, for all reliability functions $h(p)$ belonging to a certain class $\mathcal{X}$, one can prove an inequality of the form

(2.1.1) $h'(p) \geq \Psi(p,h)$, for all $0 \leq p \leq 1$, $0 \leq h \leq 1$.

This means that there exists a function $\Psi(p, h)$ on the unit square $0 \leq p \leq 1$, $0 \leq h \leq 1$, such that when a curve representing $h(p) \varepsilon \mathcal{X}$ passes through a point (p, h) then the slope of that curve at that point must be at least $\Psi(p, h)$.

If the inequality is replaced by equality, then (2.1.1) becomes a differential equation

(2.1.2) $\chi' = (p, \chi)$

which under very general assumptions on $\Psi(p, \chi)$ has a one-parametric family of solutions $\chi_c(p)$ with the parameter c. We shall say that the family of curves representing the functions $\chi_c(p)$ for $0 \leq p \leq 1$ forms a <u>grid for the class</u> $\mathcal{X}$. In view of (2.1.1), this grid has the following properties:

1° Through every point (p, h) in the unit square goes exactly one grid curve $\chi_c(p)$

2° if the curve representing a reliability function $h(p) \varepsilon \mathcal{X}$

goes through a point (p, h) and $\chi_c(p)$ is the grid curve going through the same point, then $h'(p) \geq \chi'_c(p) = \Psi(p, h)$. This means that if the curve $h(p) \in \mathcal{H}$ intersects any grid curve, then it intersects it from below. It should be noted, however, that there may be points in the unit square $0 \leq p \leq 1$, $0 \leq h \leq 1$ such that no curve $h(p) \in \mathcal{H}$ goes through them.

The knowledge of a grid may be utilized in various ways which will be discussed later, but the most immediate application is the following:

Assume that all one knows about a reliability function $h(p) \in \mathcal{H}$ is that for given component reliability p_o it assumes a known value $h(p_o) = h_o$. Then there exists a parameter value c_o such that $\chi_{c_o}(p_o) = h_o$, and from grid property 2° it follows that

$$h(p) \geq \chi_c(p) \quad \text{for all} \quad p \geq p_o .$$

2.2 It is well known that for the class $\mathcal{H}_c$ consisting of reliability functions for all coherent structures the inequality

$$(2.2.1) \qquad h'(p) \geq \frac{h(p)\,[1 - h(p)]}{p(1 - p)}$$

holds for $0 \leq p \leq 1$. This inequality, obtained for two-terminal networks in [3] and generalized in [1] to all coherent systems and in [4] to the case of components with unequal reliabilities, is of the form (2.1.1). The corresponding

differential equation of the form (2.1.2) is

$$(2.2.2) \qquad \chi' = \frac{\chi(1-\chi)}{p(1-p)} .$$

Its general integral is

$$(2.2.3) \qquad \frac{\chi_c(p)}{1-\chi_c(p)} = c\,\frac{p}{1-p} \quad , c > 0 \; , \; 0 \leq p \leq 1$$

and this one-parameter family of curves forms the so-called Moore-Shannon grid. Fig. 1 indicates the shape of the curves of this family which, for $c = 1$, includes the diagonal $\chi_1(p) = p$.

2.3 We shall need the following concepts defined by analogy with terms used in the theory of circuits: the length ℓ_Φ of a system Φ is the smallest number of components such that if only they function, the structure functions; the width w_Φ of Φ is the smallest number of components such that if only they fail, the structure fails.

According to these definitions we have

$$(2.3.1) \qquad \begin{aligned} &\Phi(\underline{x}) = 0 \quad \text{for all } \underline{x} \text{ such that } S(\underline{x}) \leq \ell_\Phi - 1 \\ &\Phi(\underline{x}) = 0 \quad \text{for some } \underline{x} \text{ such that } \ell \leq S(\underline{x}) \leq n - w \\ &\Phi(\underline{x}) = 1 \quad \text{for some } \underline{x} \text{ such that } \ell \leq S(\underline{x}) \leq n - w \\ &\Phi(\underline{x}) = 1 \quad \text{for all } \underline{x} \text{ such that } n - w + 1 \leq S(\underline{x}) . \end{aligned}$$

Sometimes the only information one has about a system Φ consists of the knowledge of ℓ_Φ, or w_Φ and, possibly, A_ℓ or A_w. The remainder of this section will be devoted to the problem of obtaining grids when some or all of the parameters ℓ, w, A_ℓ, A_w are known.

According to (2.3.1) we compute

$$E\left\{\Phi(\underline{X})\ [S(\underline{X})-\gamma]\right\} = \sum_{j=0}^{n}\ \sum_{S(\underline{x})=j} \Phi(\underline{x})\ (j-\gamma)\ P\{\underline{X}=\underline{x}\} =$$

$$\sum_{j=\gamma}^{n-w} (j-\gamma) \sum_{S(\underline{x})=j} \Phi(\underline{x})\ p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} (j-\gamma) \sum_{S(\underline{x})=j} p^j(1-p)^{n-j} =$$

$$\sum_{j=\gamma}^{n-w} (j-\gamma)\ A_j\ p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} (j-\gamma) \binom{n}{j} p^j(1-p)^{n-j}\ ,$$

hence, using formula (6.3) in [4] for $p_1 = p_2 = \cdots = p_n$,

we obtain

$$\mathrm{cov}\left\{\Phi(\underline{X}),\ S(\underline{X})\right\} = p(1-p)\ h'(p) = E\left\{\Phi(\underline{X})\ S(\underline{X})\right\} -$$

$$E\left\{\Phi(\underline{X})\right\}\ E\left\{S(\underline{X})\right\} =$$

$$\gamma\ h(p) + \sum_{j=\gamma}^{n-w} (j-\gamma)\ A_j\ p^j(1-p)^{n-j} +$$

$$\sum_{j=n-w+1}^{n} (j-\gamma) \binom{n}{j}\ p^j(1-p)^{n-j} - h(p)\ np =$$

$$(\gamma - np)\ h(p) + \sum_{j=\gamma}^{n-w} (j-\gamma)\ A_j\ p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} (j-\gamma) \binom{n}{j}\ p^j(1-p)^{n-j}.$$

Using the inequality

$$A_j \geq 1 \quad \text{for} \quad \gamma \leq j \leq n-w$$

which follows from (2.3.1), and inequality (1.9.1), one obtains

$$A_j \geq \mathrm{Max}\left\{A_\gamma \frac{\binom{n}{j}}{\binom{n}{\gamma}},\ 1\right\}, \quad \text{for} \quad \gamma \leq j \leq n-w\ .$$

Since

$$\frac{\binom{n}{j}}{\binom{n}{\gamma}} = \frac{\gamma\,!\,(n-\gamma)\,!}{j\,!\,(n-j)\,!} = \frac{(n-j+1)\ (n-j+2)\ \cdots\ (n-\gamma)}{(\gamma+1)\ (\gamma+2)\ \cdots\ j}$$

and $n-j \geq \gamma$ implies $\frac{n-j+1}{\gamma+1} > \frac{n-j+2}{\gamma+2} > \cdots > \frac{n-\gamma}{j} \geq 1$, hence

$$\frac{\binom{n}{j}}{\binom{n}{\gamma}} > 1$$

and $n-j < \gamma$ implies $\frac{n-j+1}{\gamma+1} < \frac{n-j+2}{\gamma+2} < \dots < \frac{n-\gamma}{j} < 1$,

hence $\frac{\binom{n}{j}}{\binom{n}{\gamma}} < 1$

we conclude

$$A_j > A_\gamma \frac{\binom{n}{j}}{\binom{n}{\gamma}} \quad \text{for} \quad j \leq n-\gamma$$

$$A_j \geq 1 \quad \text{for} \quad j > n-\gamma$$

and finally we obtain the inequality

$$\begin{aligned} p(1-p)\, h'(p) \geq {} & (\gamma-np)\, h(p) + \\ & \frac{A_\gamma}{\binom{n}{\gamma}} \sum_{j=\gamma}^{n-\gamma} (j-\gamma) \binom{n}{j} p^j (\gamma-p)^{n-j} + \\ & \sum_{j=n-\gamma+1}^{n-w} (j-\gamma)\, p^j (1-p)^{n-j} + \\ & \sum_{j=n-w+1}^{n} (j-\gamma) \binom{n}{j} p^j (1-p)^{n-j} \, . \end{aligned} \tag{2.3.3}$$

In this inequality, the first sum may be empty if $\gamma > n-\gamma$, and the second sum may be empty if $w > \gamma - 1$. Both these sums are empty when $\gamma > n-\gamma$ and $w > \gamma-1$ which implies $\gamma + w \geq n$, so that, in view of the known inequality $1 + w \leq n + 1$, one then has $\gamma + w = n$ or $\gamma + w = n + 1$.

Inequality (2.3.3) is of the form (2.1.1). The corresponding differential equation of the form (2.1.2) is

$$\chi' = \frac{\chi - np}{p(1-p)} + \frac{A_\chi}{\binom{n}{\chi}} \sum_{j=\chi+1}^{n-\chi} (j-\chi) \binom{n}{j} p^{j-1}(1-p)^{n-j-1} +$$

$$\sum_{j=n-\chi+1}^{n-w} (j-1)\, p^{j-1}(1-p)^{n-j-1} + \tag{2.3.4}$$

$$\sum_{j=n-w+1}^{n} (j-\chi) \binom{n}{j} p^{j-1}(1-p)^{n-j}$$

and, again, the first or the second sum or both cam be empty.

The general solution of (2.3.4) is

$$\chi_c = cp^\chi (1-p)^{n-\chi} + A_\chi \sum_{j=\chi+1}^{n-\chi} \frac{\binom{n}{j}}{\binom{n}{\chi}} p^j(1-p)^{n-j} +$$

$$\sum_{j=n-\chi+1}^{n-w} p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} \binom{n}{j} p^j(1-p)^{n-j} . \tag{2.3.5}$$

The family of functions (2.3.5) constitutes a grid for the class of reliability functions corresponding to coherent systems with given n, χ, w and A_χ. This class will be denoted by $\mathscr{X}_c(n, \chi, w, A_\chi)$.

If χ and w are known, but A_χ is not known, then (2.3.3) can be replaced by a (weaker) inequality by setting $A_\chi = 1$, and one obtains the grid

$$\chi_c = cp^\chi(1-p)^{n-\chi} + \sum_{j=\chi+1}^{n-\chi} \frac{\binom{n}{j}}{\binom{n}{\chi}} p^j(1-p)^{n-j} +$$

$$\sum_{j=n-\chi+1}^{n-w} p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} \binom{n}{j} p^j(1-p)^{n-j} \tag{2.3.6}$$

for the class $\mathcal{H}_c(n, \gamma\ w)$ of reliability functions corresponding to coherent systems with given n, γ, and w.

If only n and γ are known, then the resulting grid is

$$\chi_c = cp^{\gamma}(1-p)^{n-\gamma} + \sum_{j=\gamma+1}^{n-\gamma} \frac{\binom{n}{j}}{\binom{n}{\gamma}} p^j(1-p)^{n-j} + \sum_{j=n-\gamma+1}^{n} p^j(1-p)^{n-j} \tag{2.3.7}$$

When in (2.3.3) all terms on the right side are omitted, but the first, one arrives at the particularly simple grid for $\mathcal{H}_c(n,\gamma)$:

$$\chi_c = cp^{\gamma}(1-p)^{n-\gamma} . \tag{2.3.8}$$

For $h(p) \in \mathcal{H}_c(n, \gamma, w, A_\gamma)$ one always has

$$h(p) = A_\gamma p^{\gamma}(1-p)^{n-\gamma} + \sum_{j=\gamma+1}^{n} A_j p^j(1-p)^{n-j} > A_\gamma p^{\gamma}(1-p)^{n-\gamma}$$

so that no $h(p)$ in this class can go through points in the region $h \leq A_\gamma p^{\gamma}(1-p)^{n-\gamma}$.

2.4 Again using (2.3.1) one computes

$$E\left\{[1-\phi(\underline{X})]\,[n-w-S(\underline{X})]\right\} = \sum_{j=0}^{n} \sum_{S(\underline{x})=j} [1-\phi(\underline{x})]\,(n-w-j)\, P\left\{\underline{X}=\underline{x}\right\} =$$

$$\sum_{j=0}^{\gamma-1} (n-w-j)\binom{n}{j} p^j(1-p)^{n-j} + \sum_{j=\gamma}^{n-w-1} (n-w-j)\, A^*_j\, p^j(1-p)^{n-j}$$

where $A^*_j = \binom{n}{j} - A_j$ = number of cuts of size j . Hence

$$\operatorname{cov}\left\{\phi(\underline{X}), S(\underline{X})\right\} = p(1-p)\,h'(p) = E\left\{\phi(\underline{X})\,S(\underline{X})\right\} - E\left\{\phi(\underline{X})\right\}E\left\{S(\underline{X})\right\} =$$

$$[1-h(p)]\,[np-(n-w)] + \sum_{j=0}^{\gamma-1} (n-w-j)\binom{n}{j} p^j(1-p)^{n-j} + \sum_{j=\gamma}^{n-w-1} (n-w-j)\, A^*_j\, p^j(1-p)^{n-j}$$

and duplicating the arguments of Section 2.3 one obtains

$$p(1-p)\,h'(p) \geq [1-h(p)]\,[np-(n-w)] +$$

$$\sum_{j=0}^{\chi-1} (n-w-j)\binom{n}{j} p^j(1-p)^{n-j} +$$

(2.4.1)

$$\sum_{j=\chi}^{w-1} (n-w-j)\, p^j(1-p)^{n-j} +$$

$$\frac{A^*_{n-w}}{\binom{n}{w}} \sum_{j=w}^{n-w-1} (n-w-j)\binom{n}{j} p^j(1-p)^{n-j} ,$$

an inequality of the form (2.1.1). As was done in Section 2.3 with regard to inequality (2.3.3), we may retain all or some terms of the right side of (2.4.1), replace in each case inequality by equality, integrate the resulting differential equations, and obtain grids for the respective classes of reliability functions. We consider here explicitly only the case when all but the first term on the right side of (2.4.1) are omitted. One obtains then the simple grid

(2.4.2) $$\chi_c(p) = 1 - cp^{n-w}(1-p)^w$$

for the class $\mathcal{H}_c(n,w)$ of reliability functions for coherent systems for which n and w are known.

2.5 <u>The use of several grids for the same class of reliability functions</u>.

2.5.1 If there are several different grids for a given class $\mathcal{H}$ of reliability functions, then all these grids can be used to obtain lower bounds for an $h(p) \in \mathcal{H}$ which are better

than bounds based on any single of the grids. For example, let us consider a class $\mathcal{H}$ for which there are two grids $\chi_a(p)$ and $\lambda_b(p)$, with parameters a and b, respectively. If for an $h(p) \varepsilon \mathcal{H}$ it is known that $h(p_0) = h_0$, then one can determine a_0 and b_0 so that $\chi_{a_0}(p_0) = \lambda_{b_0}(p_0) = h_0$, and choose that one of the curves $\chi_{a_0}(p)$, $\lambda_{b_0}(p)$ which is steeper at p_0. If, say, $\chi'_{a_0}(p_0) > \lambda'_{b_0}(p_0)$ then $\chi_{a_0}(p_0)$ will be used as a lower bound for $h(p)$ for $p \geq p_0$ until, possibly, it intersects some curve of the λ-grid which is steeper at the point of intersection, i.e., until the first value $p_1 > p_0$ such that for some b_1 one has $\lambda_{b_1}(p_1) = \chi_{a_0}(p_1)$, $\lambda'_{b_1}(p_1) > \chi'_{a_0}(p_1)$. Then $\lambda_{b_1}(p)$ can be used as a lower bound for $p \geq p_1$, etc. Fig. 2 illustrates this procedure.

In some cases an analytic discussion can be carried out for the use of several grids, and a practically useful example of such a discussion follows.

2.5.2 In sections 2.2, 2.3, and 2.4 we have seen that the families of curves (2.3.8), (2.2.3) and (2.4.2) are grids for the class $\mathcal{H}_c(n, \chi, w)$. For the purposes of our discussion we rewrite the equations of these grids

(2.5.2.1) $$\chi_a(p) = ap^{\chi}(1-p)^{n-\chi}$$

(2.5.2.2) $$\lambda_b(p) = \frac{b\ p}{1 + (b-1)p}$$

(2.5.2.3) $$\mu_c(p) = 1 - cp^{n-w}(1-p)^w .$$

In order that the curve of each of these families passes through a given point (p_1, h_1) the corresponding parameters must assume the values

$$a_1 = \frac{h_1}{p_1^{\gamma}(1-p_1)^{n-\gamma}}$$

$$b_1 = \frac{h_1}{p_1} \cdot \frac{1-p_1}{1-h_1}$$

$$c_1 = \frac{1-h_1}{p_1^{n-w}(1-p_1)^{w}} .$$

The derivatives of the three curves passing through (p_1, h_1) at that point are

$$\chi'_{a_1}(p_1) = \frac{h_1(\gamma-np_1)}{p_1(1-p_1)} \tag{2.5.2.4}$$

$$\lambda'_{b_1}(p_1) = \frac{h_1(1-h_1)}{p_1(1-p_1)} \tag{2.5.2.5}$$

$$\mu'_{c_1}(p_1) = \frac{(1-h_1)(w-n+np_1)}{p_1(1-p_1)} . \tag{2.5.2.6}$$

We have

$$\chi'_{a_1}(p_1) > 0 \quad \text{if and only if} \quad 0 < p_1 < \frac{\gamma}{n}$$

$$\lambda'_{b_1}(p_1) > 0 \quad \text{for all } p_1 , \quad 0 < p_1 < 1$$

$$\mu'_{c_1}(p_1) > 0 \quad \text{if and only if} \quad \frac{n-w}{n} < p_1 < 1$$

so that (2.5.2.1) is a non-trivial grid only for $0 < p < \frac{\gamma}{n}$ (2.5.2.3) only for $\frac{n-w}{n} < p_1 < 1$, while the Moore-Shannon

grid (2.5.2.2) consists of functions which increase for all p and hence is useful for $0 < p < 1$.

In view of the known inequality $\chi + w \leq n + 1$, we have $\frac{\chi}{n} \leq \frac{n-w}{n}$, except for the case when $\chi + w = n + 1$ which occurs if and only if the structure is " χ out of n "; in this case $h(p) = \sum_{j=\chi}^{n} \binom{n}{j} p^j (1-p)^{n-j}$ and is completely known. In all other cases the intervals $(0, \frac{\chi}{n})$, $(\frac{\chi}{n}, \frac{n-w}{n})$, $(\frac{n-w}{n}, 1)$ are non-overlapping, and for $0 < p < \frac{\chi}{n}$ we need to consider only grids (2.5.2.1) and (2.5.2.2), for $\frac{\chi}{n} < p < \frac{n-w}{n}$ only grid (2.5.2.2), and for $\frac{n-w}{n} < p < 1$ only grids (2.5.2.2) and (2.5.2.3). Comparing the derivatives (2.5.2.4) and (2.5.2.5) we see that the χ-grid is steeper for $0 < p < \frac{\chi}{n}$ if and only if $h > np - \chi + 1$, and comparing (2.5.2.5) and (2.5.2.6) we find that the μ-grid is steeper for $\frac{n-w}{n} < p < 1$ if and only if $h > np - n + w$. The parallel lines $h = np - \chi + 1$ and $h = np - n + w$ divide therefore the unit square in three regions, from left to right, such that the χ-grid is steepest at all points of the first region, the λ-grid in the second and the μ-grid in the third region.

2.5.3 A specific example is presented in Figure 3. For $n = 10$, $\chi = 5$, $w = 2$, the lines

$$(\chi_1) \quad h = 10p - 4$$

$$(\chi_2) \quad h = 10p - 8$$

partition the unit square in the three regions described in the preceding section. The curves of the Moore-Shannon grid (2.5.2.2), shown before in Figure 1, are reproduced in Figure 3 in solid lines and, in addition, several curves of the χ-grid (2.5.2.1) are indicated by dotted lines, and of the μ-grid (2.5.2.3) by broken lines.

If it is known that $h(p) \varepsilon \mathcal{H}_c$ ($n = 10$, $\gamma = 5$, $w = 2$) and that the graph of $h(p)$ goes through the point $P_1 =$ $(.20, .05)$, then our theory tells us that for $p \geq .20$ that graph is bounded from below by a curve which goes along the χ-curve through P_1 to its intersection with γ_1, then along the Moore-Shannon curve (in this case the diagonal) to its intersection with γ_2, then along the μ-curve. This lower bound is indicated by a heavy line.

Similarly, if a reliability function $h(p)$ of our family is known to go through $P_2 = (.34, .10)$ then for $p \geq .34$ one obtains for $h(p)$ the lower bound indicated by the heavy line beginning at P_2.

Another lower bound for $h(p)$ going through $P_3 = (.32, .40)$ is indicated by the heavy line beginning at that point.

<u>Addendum</u>

It should be mentioned that an improvement of the Moore-Shannon grid has been recently obtained. One can show that the following inequalities hold for all $h(p) \varepsilon \mathcal{H}_c$

$$(-p \log p)\, h'(p) \geq -h \log h$$

$$[-(1-p) \log (1-p)]\, h'(p) \geq -(1-h) \log (1-h).$$

The corresponding grids are

(i) $\chi_c(p) = p^c \quad , c > 0$

(ii) $\chi_c(p) = 1 - (1-p)^c \; , \quad c > 0 .$

These grids are an improvement on (2.2.3), since (i) is steeper than the corresponding Moore-Shannon curve at every point such that $p > h$, and (ii) at every point such that $p < h$. A derivation of this new grid is being prepared for publication [5].

REFERENCES

[1] Z. W. Birnbaum, J. D. Esary, and S. C. Saunders, "Multi Component Systems and Structures and Their Reliabilities", Technometrics, Vol. 3, No. 1, 1961, pp. 55-77.

[2] J. D. Esary and A. W. Marshall, "System Structure and the Existence of a System Life", Technometrics, Vol. 6, No. 4, pp. 459-462.

[3] E. F. Moore and C. E. Shannon, "Reliable Circuits Using Less Reliable Relays", Journal of the Franklin Institute, Vol. 262, 1956, pp. 191-208 and pp. 281-297.

[4] J. D. Esary and F. Proschan, "Coherent Structures of Non-Identical Components", Technometrics, Vol. 5, 1963, pp. 191-209.

[5] Z. W. Birnbaum, J. D. Esary, and A. W. Marshall, "Stochastic Characterization of Wear-Out for Components and Systems", Boeing Scientific Research Laboratories Document DI-8L-0460, in preparation.

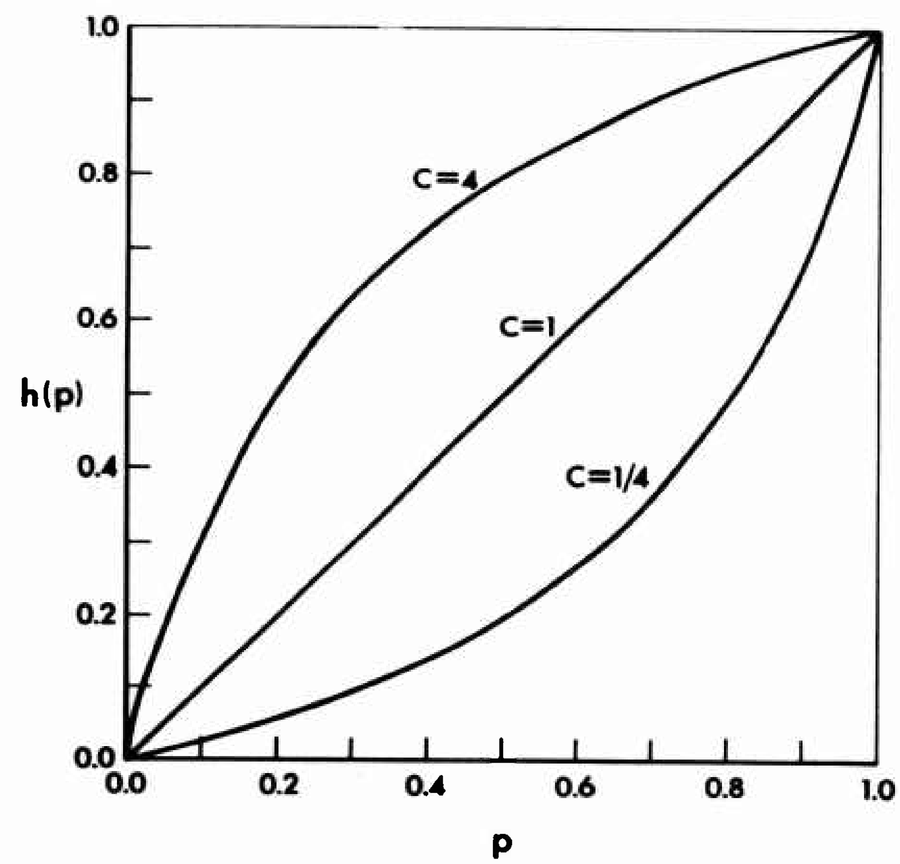

FIGURE 1

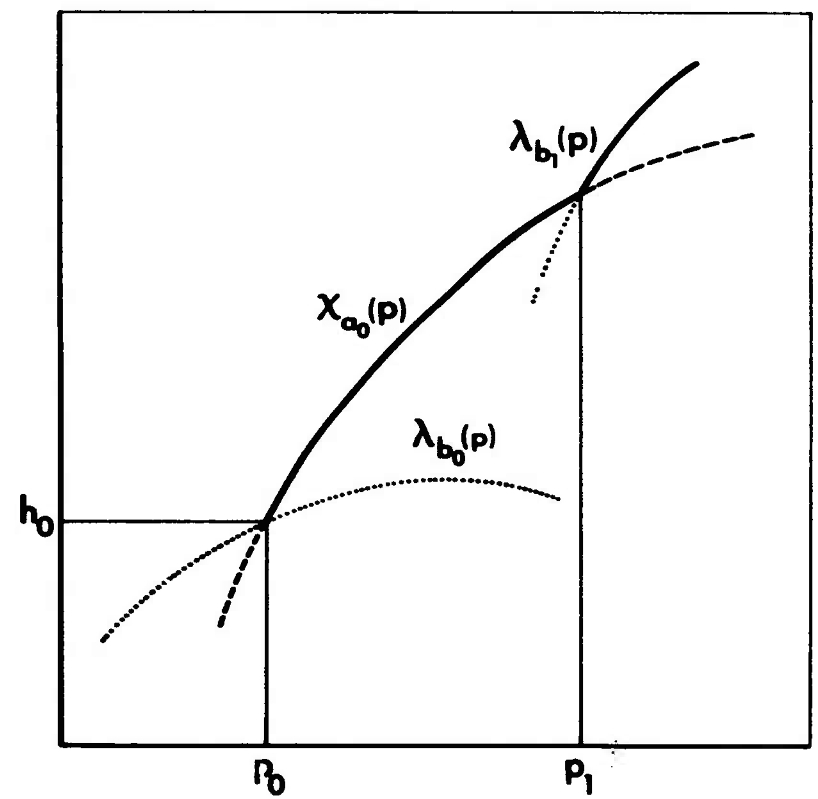

FIGURE 2

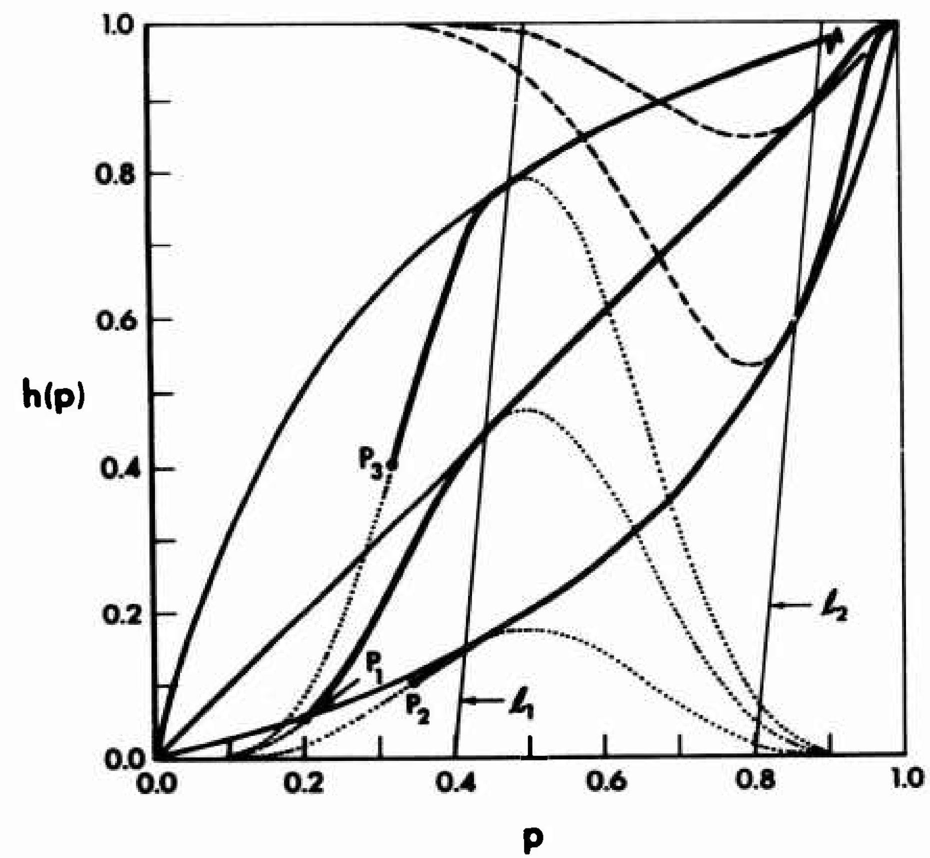

FIGURE 3

Security Classification

DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) Laboratory of Statistical Research Department of Mathematics University of Wash., Seattle, Wash., 98105	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP	
3. REPORT TITLE Some Inequalities for Reliability Functions		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report No. 44, August, 1965		
5. AUTHOR(S) (Last name, first name, initial) Birnbaum, Z. W. and Esary, J. D.		
6. REPORT DATE August 18, 1965	7a. TOTAL NO. OF PAGES 19	7b. NO. OF REFS 5
8a. CONTRACT OR GRANT NO. ONR-477(11) b. PROJECT NO. NR 042038 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 44 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. AVAILABILITY/LIMITATION NOTICES Qualified requesters may obtain copies from DDC		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY U. S. Navy Office of Naval Research Washington, D. C.	

13. ABSTRACT

A method is presented for obtaining lower bounds for $h'(p)$, the derivative of the reliability function of a coherent binary structure with binary components, when only partial information about this structure is available. In particular, the case is studied when only the number of components, the length and the width of the structure are known.

DD FORM 1 JAN 64 1473

Security Classification

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
reliability coherent systems						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ______________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

Security Classification